13.1) Let $X$ be a topological space and let $A$ be a subset of $X$. Suppose that for each $x \in A$ there is an open set $U$ containing $x$ such that $U \subseteq A$. Show that $A$ is open in $X$.

Proof:
Since for each $x \in A$ there exist $U_x \in \mathcal{T}$ such that $x \in U_x$ and $U_x \subseteq A$, we have $A = \bigcup_{x \in A} U_x$. That is, $A$ is the union of open sets in $X$ and thus by definition of the topology on $X$, $A$ is open.

13.3) Show that the collection $\mathcal{T}_c$ given in example 4 (pg 77) is a topology on the set $X$. Is the collection $\mathcal{T}_\infty = \{ U | X - U$ is infinite or empty or all of $X \}$ a topology on $X$.

Proof: PART (1)
From example 4 $\mathcal{T}_c = \{ U | X - U$ is countable or is all of $X \}$. To show that $\mathcal{T}_c$ is a topology on $X$, we must show we meet all 3 conditions of the definition of topology.
First, since $X - X = \emptyset$ is countable and $X - \emptyset = X$ is all of $X$ we have $X$ and $\emptyset \in \mathcal{T}_c$.
Second, let $U_\alpha \in \mathcal{T}_c$ for $\alpha \in I$. We must look at two cases: (1) $X - U_\alpha = X$ for all $\alpha$ and (2) There exist $\alpha$ such that $X - U_\alpha$ is countable.

Case (1): $X - U_\alpha = X$
This implies $U_\alpha = \emptyset$ for all $\alpha$. So $X - (\bigcup_{\alpha \in I} U_\alpha) = \bigcap_{\alpha \in I} (X - U_\alpha) = X$ and thus, $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_c$.

Case (2): There exist $\alpha$ such that $X - U_\alpha$ is countable.
We have $X - (\bigcup_{\alpha \in I} U_\alpha) = \bigcap_{\alpha \in I} (X - U_\alpha) \subseteq X - U_\alpha$ which is countable. Thus, since subsets of countable sets are countable, $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_c$.
Third, we must show finite intersections are in $\mathcal{T}_c$. Let $U_\alpha \in \mathcal{T}_c$ for $1 \leq \alpha \leq n$. Again, we must check two cases.

Case (1): There exists an $\alpha$ such that $X - U_\alpha = X$.
So $X - (\bigcap_{\alpha = 1}^n U_\alpha) = \bigcup_{\alpha = 1}^n (X - U_\alpha) = X$ and thus, $\bigcap_{\alpha = 1}^n U_\alpha \in \mathcal{T}_c$.

Case (2): $X - U_\alpha$ is countable for all $\alpha$.
We have $X - (\bigcap_{\alpha = 1}^n U_\alpha) = \bigcup_{\alpha = 1}^n (X - U_\alpha)$.

Since a finite collection of countable sets is countable, we again have $\bigcap_{\alpha = 1}^n U_\alpha \in \mathcal{T}_c$.
Thus $\mathcal{T}_c$ is a topology on $X$. ■
Proof: PART (2)
The collection $\mathcal{T}_\infty = \{U | X - U \text{ is infinite or empty or all of } X \}$ is NOT a topology on $X$.

By counterexample let $X = \mathbb{R}$ and let $U_1 = (-\infty, 0)$ and $U_2 = (0, \infty)$. $U_1$ and $U_2$ are in $\mathcal{T}_\infty$ since $X - U_1 = [0, \infty)$ and $X - U_2 = (-\infty, 0]$ are both infinite. Condition 2 of the definition of topology fails. That is, $U_1 \cup U_2 \notin \mathcal{T}_\infty$ since $X - (U_1 \cup U_2) = \{0\}$.

13.4a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on $X$, show that $\bigcap \mathcal{T}_\alpha$ is a topology on $X$. Is $\bigcup \mathcal{T}_\alpha$ a topology on $X$?

13.4b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on $X$. Show that there is a unique smallest topology on $X$ containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

13.4c) If $X = \{a, b, c\}$, let $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$.

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Proof: 13.4a.1
Since each $\mathcal{T}_\alpha$ is a topology, $X$ and $\emptyset \in \bigcap \mathcal{T}_\alpha$. For arbitrary unions, let $U_\varphi \in \bigcap \mathcal{T}_\alpha$ for $\varphi \in I$. Then $U_\varphi \in \mathcal{T}_\alpha$ for all $\alpha$ and thus $\bigcup U_\varphi \in \bigcap \mathcal{T}_\alpha$. Finally, for finite intersections, let $U_j \in \bigcap \mathcal{T}_\alpha$ for $1 \leq j \leq n$. So $\bigcap_{j=1}^n U_j \in \mathcal{T}_\alpha$ for all $\alpha$ and hence $\bigcap_{j=1}^n U_j \in \bigcap \mathcal{T}_\alpha$. Thus $\bigcap \mathcal{T}_\alpha$ is a topology on $X$.

Proof: 13.4a.2
$\bigcup \mathcal{T}_\alpha$ is NOT a topology on $X$. For $U_1 \in \mathcal{T}_\varphi$ and $U_2 \in \mathcal{T}_\beta$, $U_1 \cap U_2$ are not necessarily in $\bigcup \mathcal{T}_\alpha$. As a counter example, let $X = \{a, b, c\}$, $\mathcal{T}_\varphi = \{X, \emptyset, \{a, b\}\}$, $\mathcal{T}_\beta = \{X, \emptyset, \{b, c\}\}$, $U_1 = \{a, b\}$, and $U_2 = \{b, c\}$. Then $\bigcup \mathcal{T}_\alpha = \mathcal{T}_\varphi \cup \mathcal{T}_\beta$ but, $U_1 \cap U_2 = \{b\} \notin \bigcup \mathcal{T}_\alpha$.

Proof: 13.4b.1

For existence, let $\mathcal{T} = \bigcap \mathcal{T}_\beta$, where $\{\mathcal{T}_\beta\}$ is the collection of all topologies containing $\bigcup \mathcal{T}_\alpha$. We claim that $\mathcal{T} = \bigcap \mathcal{T}_\beta$ is the uniquely smallest topology on $X$ containing all the collections $\mathcal{T}_\alpha$. First, by part a, we know that $\mathcal{T}$ is a topology and $\mathcal{T}_\beta$ contains $\bigcup \mathcal{T}_\alpha$ for all $\beta$. Since $\mathcal{T}$ is the intersection of all topologies containing $\bigcup \mathcal{T}_\alpha$, it must be the smallest.

For uniqueness, assume there exists a topology $\mathcal{T}'$ that contains $\bigcup \mathcal{T}_\alpha$ and is such that $\mathcal{T}' \subseteq \mathcal{T}$. Since $\mathcal{T}$ is the smallest, we must have $\mathcal{T} \subseteq \mathcal{T}'$. Thus $\mathcal{T}' = \mathcal{T}$ as desired and $\mathcal{T} = \bigcap \mathcal{T}_\beta$ is the uniquely smallest topology on $X$ containing all the collections $\mathcal{T}_\alpha$. 

\[\copyright\ Steven T Riggs 2014\]
Compiled: 09/15/2014 7:34pm

Page 2 of 6
Proof: 13.4b.2
As for the largest topology contained in all $\mathcal{T}_\alpha$, consider $\mathcal{T} = \bigcap \mathcal{T}_\alpha$. It is contained in $\mathcal{T}_\alpha$ for all alpha and, by part a, is a topology. Since it is the intersection of all $\mathcal{T}_\alpha$, it must be the largest. Now for uniqueness, assume there is another topology $\mathcal{T}'$ contained in each $\mathcal{T}_\alpha$ and such that $\mathcal{T}' \supseteq \mathcal{T}$. Since $\mathcal{T}$ is the largest, we must have $\mathcal{T} \supseteq \mathcal{T}'$. Thus, again, $\mathcal{T} = \mathcal{T}$ as desired and $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ is the uniquely largest topology on $X$ contained in all $\mathcal{T}_\alpha$.

Prof: 13.4c
The smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. The largest topology contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\{\emptyset, X, \{a\}\}$.

13.5) Show that if $\mathcal{A}$ is a basis for a topology on $X$, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on $X$ that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Proof: PART (1)
Let $\mathcal{T}_\mathcal{A}$ be the topology generated by the basis $\mathcal{A}$ and let $\{\mathcal{T}_{\mathcal{A}_\alpha}\}$ be the collection of all topologies containing $\mathcal{A}$.
First, $\bigcap \mathcal{T}_{\mathcal{A}_\alpha} \subseteq \mathcal{T}_\mathcal{A}$ since $\mathcal{T}_\mathcal{A} \in \{\mathcal{T}_{\mathcal{A}_\alpha}\}$.
For the reverse inclusion let $U \in \mathcal{T}_\mathcal{A}$. Then for all $x \in U$ we have $A_x \in \mathcal{A}$ such that $A_x \subseteq U$ and $U = \bigcup A_x$. Now since $\mathcal{T}_{\mathcal{A}_\alpha}$ contains $\mathcal{A}$ for all alpha, $U = \bigcup_{x \in U} A_x$.
That is, $\mathcal{T}_\mathcal{A} \subseteq \bigcap \mathcal{T}_{\mathcal{A}_\alpha}$ and $\mathcal{T}_\mathcal{A} = \bigcap \mathcal{T}_{\mathcal{A}_\alpha}$ as desired.

Prof: PART (2)
Let $\mathcal{T}_\mathcal{A}$ be the topology generated by the subbasis $\mathcal{A}$ and let $\{\mathcal{T}_{\mathcal{A}_\alpha}\}$ be the collection of all topologies containing $\mathcal{A}$. Then as above, $\bigcap \mathcal{T}_{\mathcal{A}_\alpha} \subseteq \mathcal{T}_\mathcal{A}$ since $\mathcal{T}_\mathcal{A} \in \{\mathcal{T}_{\mathcal{A}_\alpha}\}$.
Conversely, let $U \in \mathcal{T}_\mathcal{A}$. So $U = \bigcup_{\alpha \in I} \bigcap_{i=1}^{\alpha} A_i$, where $A_{\alpha_1}, \ldots, A_{\alpha_n} \in \mathcal{A}$. Since $A_{\alpha_1}, \ldots, A_{\alpha_n} \in \bigcap \mathcal{T}_{\mathcal{A}_\alpha}$ and $\bigcap \mathcal{T}_{\mathcal{A}_\alpha}$ is a topology, we have $\bigcap_{i=1}^{\alpha} A_i \in \bigcap \mathcal{T}_{\mathcal{A}_\alpha}$ for all $\alpha$.

Now we have $\bigcup_{\alpha \in I} \bigcap_{i=1}^{\alpha} A_i = U \in \bigcap \mathcal{T}_{\mathcal{A}_\alpha}$ and hence $\mathcal{T}_\mathcal{A} \subseteq \bigcap \mathcal{T}_{\mathcal{A}_\alpha}$.

Thus $\mathcal{T}_\mathcal{A} = \bigcap \mathcal{T}_{\mathcal{A}_\alpha}$ as desired.

13.6) Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are not comparable.

Proof:
The lower limit topology, $\mathcal{T}_{\mathbb{R}_l}$, is the topology generated on $\mathbb{R}$ by the collection of all half-open intervals of the form $[a, b) = \{x | a \leq x < b\}$.
The $K$-topology, $\mathcal{T}_{\mathbb{R}_k}$, on $\mathbb{R}$ is the topology generated by the collection of all open intervals $(a, b)$, along with the sets of the form $(a, b) - K$. Here $K$ is the is the set of all positive rationals of the form $\frac{1}{n}$, for $n \in \mathbb{Z}^+$.
We must show $\mathcal{T}_{\mathbb{R}_l} \nsubseteq \mathcal{T}_{\mathbb{R}_k}$ and $\mathcal{T}_{\mathbb{R}_k} \nsubseteq \mathcal{T}_{\mathbb{R}_l}$. That is, there exists an element $U_l \in \mathcal{T}_{\mathbb{R}_l}$ such that $U_l \notin \mathcal{T}_{\mathbb{R}_k}$ and an element $U_k \in \mathcal{T}_{\mathbb{R}_k}$ such that $U_k \notin \mathcal{T}_{\mathbb{R}_l}$.
Notice that \([0,1) \in \mathcal{T}_{\mathbb{R}_i}\). No bases element from \(\mathbb{R}_\mathcal{K}\) exists that contains 0 and is a subset of \([0,1)\). That is, there is no open interval \((a,b)\) or interval \((a,b) - \mathcal{K}\) that is a subset of \([0,1)\) and contains 0. Thus \([0,1) \notin \mathcal{T}_{\mathbb{R}_\mathcal{K}}\) and hence \(\mathcal{T}_{\mathbb{R}_i} \notin \mathcal{T}_{\mathbb{R}_\mathcal{K}}\).

Conversely, note that \((-\frac{1}{n},\frac{1}{n}) - \mathcal{K}\) is an element of \(\mathbb{R}_\mathcal{K}\). Additionally, notice for any neighborhood \([a,b]\) containing 0 that it will also contain \(\frac{1}{n}\) for some \(n \in \mathbb{N}\). That is, \(0 \in (-\frac{1}{n},\frac{1}{n}) - \mathcal{K}\) but there is no interval \([a,b]\) such that \(0 \in [a,b)\) and \([a,b) \subset (-\frac{1}{n},\frac{1}{n}) - \mathcal{K}\). Hence \((-\frac{1}{n},\frac{1}{n}) - \mathcal{K}\) \(\notin \mathcal{T}_{\mathbb{R}_i}\) and \(\mathcal{T}_{\mathbb{R}_\mathcal{K}} \notin \mathcal{T}_{\mathbb{R}_i}\). Thus the topologies of \(\mathbb{R}_i\) and \(\mathbb{R}_\mathcal{K}\) are not comparable.

\[\square\]

**16.1)** Show that if \(Y\) is a subspace of \(X\), and \(A\) is a subset of \(Y\), then the topology \(A\) inherits as a subspace of \(Y\) is the same as the topology it inherits as a subspace of \(X\).

Proof:

By the subspace topology on \(Y, U \subseteq Y\) is open in \(Y\) if \(U = V \cap Y\) for some \(V\) open in \(X\). Similarly, \(O \subseteq A\) is open in \(A\) if \(O = U \cap A\). That is, \(O = U \cap A = (V \cap Y) \cap A = V \cap A\) and the topology \(A\) inherits as a subspace of \(Y\) is the same as the topology it inherits as a subspace of \(X\).

**16.5)** Let \(X\) and \(X'\) denote a single set in the topologies \(\mathcal{T}\) and \(\mathcal{T}'\), respectively; let \(Y\) and \(Y'\) denote a single set in the topologies \(\mathcal{U}\) and \(\mathcal{U}'\), respectively. Assume these sets are nonempty.

(a) Show that if \(\mathcal{T}' \supset \mathcal{T}\) and \(\mathcal{U}' \supset \mathcal{U}\), then the product topology on \(X' \times Y'\) is finer than the product topology on \(X \times Y\).

(b) Does the converse of (a) hold? Justify your answer.

Proof: PART A

Let \(W\) be in the product topology on \(X \times Y\). Then for all \((x, y) \in W\), there exist a basis element \(T \times U\) such that \(T \times U \subseteq W\) where \(T \in \mathcal{T}\) and \(U \in \mathcal{U}\). Now since \(\mathcal{T}'\) and \(\mathcal{U}'\) are finer than \(\mathcal{T}\) and \(\mathcal{U}\) respectively, \(T\) can be written as a union of elements of \(\mathcal{T}'\) and \(U\) can be written as a union of elements of \(\mathcal{U}'\). That is, \(T\) and \(U\) are elements of \(\mathcal{T}'\) and \(\mathcal{U}'\) respectively. So \(T \times U\) is a basis element for \(X' \times Y'\) and hence is an element of the product topology on \(X' \times Y'\). Thus the product topology on \(X' \times Y'\) is finer than the product topology on \(X \times Y\).

**PART B**

The converse is as follows: If the product topology on \(X' \times Y'\) is finer than the product topology on \(X \times Y\), then \(\mathcal{T}' \supset \mathcal{T}\) and \(\mathcal{U}' \supset \mathcal{U}\).

Claim: The converse is true.

Proof: PART B

Let \(T \in \mathcal{T}\) and \(U \in \mathcal{U}\). We must show \(T \in \mathcal{T}'\) and \(U \in \mathcal{U}'\). We have that \(T \times U\) is a basis element for the topology on \(X \times Y\) and hence is open in \(X \times Y\). Since the topology on \(X' \times Y'\) is finer, \(T \times U\) is open in \(X' \times Y'\). Thus \(T \in \mathcal{T}'\) and \(U \in \mathcal{U}'\) as desired.

\[\square\]
16.8) If \( L \) is a straight line in the plane, describe the topology \( L \) inherits as a subspace of \( \mathbb{R}_l \times \mathbb{R} \) and as a subspace of \( \mathbb{R}_l \times \mathbb{R}_l \). In each case it is a familiar topology.

**PART A:** \( \mathbb{R}_l \times \mathbb{R} \)
- All vertical lines inherit the standard topology.
- All diagonal lines inherit the standard topology union the lower limit topology.
- All horizontal lines inherit the lower limit topology.

**PART B:** \( \mathbb{R}_l \times \mathbb{R}_l \)
- All vertical, horizontal, and positive sloped diagonal lines inherit the lower limit topology.
- All negative sloped diagonal lines inherit the discrete topology.

16.9) Show that the dictionary order topology on the set \( \mathbb{R} \times \mathbb{R} \) is the same as the product topology \( \mathbb{R}_d \times \mathbb{R} \), where \( \mathbb{R}_d \) denotes \( \mathbb{R} \) in the discrete topology. Compare this topology with the standard topology on \( \mathbb{R}^2 \).

**Proof:**

**PART 1**
We must show for the dictionary topology \( \mathcal{T} \) on \( \mathbb{R} \times \mathbb{R} \) and the product topology \( \mathcal{T}_d \) on \( \mathbb{R}_d \times \mathbb{R} \), as stated, we have \( \mathcal{T} \subseteq \mathcal{T}_d \) and \( \mathcal{T}_d \subseteq \mathcal{T} \). We will show that the bases element of each contains a bases element of the other. Let \( (a \times b, a \times d) \) be a bases element of \( \mathcal{T} \). Then for all \( x \in (a \times b, a \times d) \) we have \( x \) is contained in a bases element of the form \( \{a\} \times (b, d) \subset (a \times b, a \times d) \) of \( \mathcal{T}_d \). That is, \( \mathcal{T} \subseteq \mathcal{T}_d \).

Conversely, let \( \{a\} \times (b, d) \) be a basis element of \( \mathcal{T}_d \). Then for all \( x \in \{a\} \times (b, d) \) we have \( x \in (a \times b, a \times d) \subset \{a\} \times (b, d) \) of \( \mathcal{T} \). That is, \( \mathcal{T}_d \subseteq \mathcal{T} \).

Hence, \( \mathcal{T} = \mathcal{T}_d \).

**PART 2**

Claim: The topology \( \mathcal{T} \) from above is finer than the standard topology \( \mathcal{T}_s \) on \( \mathbb{R}^2 \).

Let \( (a, c) \times (b, d) \) be a bases element in \( \mathcal{T}_s \). Then for all \( x \in (a, c) \times (b, d) \) we have \( x \in (\{a\} \times (b, d) \cup \ldots \cup \{c\} \times (b, d)) \subset (a, c) \times (b, d) \). That is, bases elements of \( \mathcal{T}_s \) can be written as the union of bases elements of \( \mathcal{T} \). Thus, \( \mathcal{T}_s \subset \mathcal{T} \) as desired.

**ADDITIONAL PROBLEM(S)**

13.2) Consider the nine topologies on the set \( X = \{a, b, c\} \) indicated in example 1 (pg 76). Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.

Name each topology in example 1 by its row \( \times \) column address (i.e. \( \mathcal{T}_{rc} \)). Then we have the following:

- \( \mathcal{T}_{11} \) is the trivial topology and NOT finer than any (coarser than all others).
- \( \mathcal{T}_{12} \) is finer than \( \mathcal{T}_{11}, \mathcal{T}_{31} \)
- \( \mathcal{T}_{13} \) is finer than \( \mathcal{T}_{11}, \mathcal{T}_{21}, \mathcal{T}_{31} \)
- \( \mathcal{T}_{21} \) is finer than \( \mathcal{T}_{11} \)
\( \mathcal{T}_2 \) is finer than \( \mathcal{T}_1 
\)
\( \mathcal{T}_3 \) is finer than \( \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \)
\( \mathcal{T}_3 \) is finer than \( \mathcal{T}_1 \)
\( \mathcal{T}_3 \) is finer than \( \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \)
\( \mathcal{T}_3 \) is the discrete topology and finer than all others.